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2009 J. Phys. A: Math. Theor. 42 365203

(http://iopscience.iop.org/1751-8121/42/36/365203)

IOP PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND THEORETICAL

J. Phys. A: Math. Theor. **42** (2009) 365203 (16pp) doi:10.1088/1751-8113/42/36/365203

Two regularization strategies for an evolutional type inverse heat source problem

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Received 19 May 2009, in final form 16 July 2009
Published 19 August 2009
Online at stacks.iop.org/JPhysA/42/365203

Abstract
This paper investigates an evolutional type inverse problem of determining an unknown heat source function in heat conduction equations when the solution is known in a discrete point set. Being different from other ordinary inverse source problems which often rely on only one variable, the unknown coefficient in this paper depends not only on the space variable x, but also on time t. Two regularization strategies which are called the time semi-discrete scheme (TSDS) and the integral reconstruction scheme (IRS), respectively, are proposed to deal with such a problem. By the TSDS the inverse problem is transformed into a sequence of stationary inverse problems and the unknown heat source is reconstructed layer by layer, while the IRS is to recover the source function from the situation as a whole. Both theoretical and numerical studies are provided. Two numerical algorithms on the basis of the Landweber iteration are designed, and some typical numerical experiments are performed in this paper. The numerical results show that the proposed methods are stable and the unknown heat source is recovered very well.

PACS numbers: 44.05.+e, 44.10.+i
Mathematics Subject Classification: 35R30, 49J20

1. Introduction

The inverse heat source problems deal with the determination of the strength of the heat source in analysis such as the internal energy source, or the quantity of the energy generation in a computer chip, in a microwave heating process or in a chemical reaction process, etc.

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In this paper, we study an inverse problem of reconstructing an unknown heat source function in parabolic equations on the basis of measurements of temperatures specified at some internal points. The problem can be stated in the following form:

Problem P. Consider an initial-boundary value problem of the heat conduction equation as follows:

$$\begin{cases} u_t - a(x)u_{xx} + b(x)u_x + c(x)u = f(x,t), & (x,t) \in Q = (0,l) \times (0,T] \\ u(0,t) = u(l,t) = 0, & t \in (0,T] \\ u(x,0) = \phi(x), & x \in (0,l) \end{cases}, \tag{1.1}$$

where $a(x), b(x), c(x)$ and $\phi(x)$ are given smooth functions on interval $(0,l)$ and $f(x,t)$ is an unknown right-hand side in (1.1). Assume that an additional condition is given as follows:

$$u(x_j,t_n) = g(x_j,t_n), \qquad (x_j,t_n) \in Q, \qquad j = 1,2,\ldots,J; \qquad n = 1,2,\ldots,N, \tag{1.2}$$

where $g(x,t)$ is a known function which satisfies the homogeneous Dirichlet boundary condition, and J, N are two given constants. Determine the functions u and f satisfying (1.1)–(1.2).

The mathematical model (1.1)–(1.2) arises in various physical and engineering settings, e.g., in hydrology, material science, heat transfer and transport problems (see [1, 2, 19, 24]). If the temperature data $u(x,t)$ are given exactly in the whole domain Q, i.e., the extra condition (1.2) is given as follows:

$$u(x,t) = g(x,t), \qquad (x,t) \in Q, \tag{1.3}$$

then the unknown source function $f(x,t)$ can be derived directly from the following formula:

$$f = g_t - ag_{xx} + bg_x + cg. \tag{1.4}$$

However, this is just an ideal case. In practice, we shall consider inaccurate input data, e.g., the extra condition (1.2) is given as

$$u(x_j,t_n) \approx g(x_j,t_n), \qquad (x_j,t_n) \in Q \tag{1.5}$$

or

$$u(x_j,t_n) = g^\delta(x_j,t_n), \qquad (x_j,t_n) \in Q, \tag{1.6}$$

where δ is the upper bound for the noise level. Note that, to get $f(x,t)$ from $g(x,t)$ by (1.4), one has to compute the numerical derivatives of $g(x,t)$ with respect to x and t, particularly the second derivative with respect to x. Therefore, the inverse problem **P** is ill-posed in the sense of Hadamard (see [9, 13, 15, 20]). Moreover, the inverse problem **P** is under-determined. In fact, it is not adequate for the extra condition (1.2) to determine the unknown heat source, namely, that one cannot identify the unknown coefficient $f(x,t)$ uniquely and stably by using (1.2). Since the discrete form (1.6) is not convenient for analysis, we assume in this paper that, without loss of generality, the overspecified observation is given as

$$u(x,t) = g^\delta(x,t), \qquad (x,t) \in Q, \tag{1.7}$$

where $g^\delta(x,t)$ is a continuous function which is obtained from (1.6) by the interpolation and smoothing technique. It can easily be seen that there is no evident difference between the two forms (1.6) and (1.7), because they all contain errors and, if necessary, it is natural to transform (1.6) into (1.7).

Inverse source coefficient problems for parabolic equations are well studied in the literature (see, for instance, [3–6, 14, 21, 23]). In [3], the inverse problem of identifying the source coefficient $f(t)$ in a 1D heat equation

$$\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} = f(t), \qquad (x,t) \in Q$$

from a solution specified at internal points has been studied carefully. The reconstruction of the source term $f(x,t) = \lambda(t)\eta(x)$, where $\lambda(t)$ is the unknown coefficient to be determined, has been discussed in [21]. In [14], the inverse problem of identifying the source coefficient $f(x)$ in the following heat equation

$$u_t - \triangle u = f(x), \qquad (x,t) \in Q$$

from the final overspecified data $u(x,T)$ has been studied carefully by using the boundary element method. In [4], the author considered the determination of $f(x)$ by the spectral theory from the overspecified boundary data. For the space- and time-dependent case $f = f(x,t)$, a numerical algorithm based on the mollification method is proposed in [23] to obtain the numerical solution.

In this paper, we use an optimal control framework (see [7, 8, 22]) to seek the stable numerical solution of problem **P**. Such a problem is a natural extension of that in [14]. The unknown coefficient in [14] is purely space dependent, while in this paper it depends not only on the space variable x, but also on time t, which may occur in the case that the property of heat source varies with space and time.

We propose two numerical methods to deal with problem **P**.

The first one comes from the idea in [7], where an evolutional inverse heat conduction problem has been resolved completely. We solve problem **P** by using the so-called time semi-discrete scheme (TSDS), i.e., we find $f(x,t_n)$ step by step, where $t_n = n\tau$, and $\tau = \frac{T}{N}, n = 1, 2, \ldots, N$. In fact, if $f(x,t_0), f(x,t_1), \ldots, f(x,t_{n-1})$ have been defined, then from the given extra condition (1.7)

$$u(x,t_n) = g^\delta(x,t_n),$$

we find $f(x,t_n)$ such that

$$J_n(f(\cdot,t_n)) = \inf_{f\in\mathcal{A}} J_n(f),$$

where $\mathcal{A}$ is an appropriate admissible set and J_n is a control functional. Therefore, we obtain an approximate function $\tilde{f}(x,t)$ defined as follows:

$$\tilde{f}(x,t) = \begin{cases} f(x,t_n), & t = t_n, \\ \text{linear}, & t_{n-1} \leqslant t \leqslant t_n. \end{cases}$$

In the sense of numerical computation, $\tilde{f}(x,t)$ can be taken as an approximate solution of $f(x,t)$ provided that τ is small enough.

By the TSDS, the original evolutional type inverse source problem is transformed into N inverse source problems which are purely spatially dependent. In other words, to gain the numerical solution of the original problem, we shall treat N stationary inverse source problems. Undoubtedly, this method is stable since the stability is based on those of the stationary cases which have been proved in [14]. Noticing that such a method is first proposed to deal with nonlinear inverse problems, we feel uncertain whether we should use it for linear inverse problems. However, it can easily be seen from (1.1) that as $\phi = 0$ the extra condition g is linearly dependent on the unknown source f, i.e., the problem **P** is indeed a linear inverse problem (if $\phi \neq 0$, we can use a variable transformation to remove it). It is well known that the Landweber iteration method (see [15]) is a classical tool to deal with linear inverse problems. Moreover, for the Landweber iteration it is not required to construct the regularization term or to choose the regularization parameter. In fact, the number of Landweber iterations is indeed the regularization parameter for the iterative scheme, while the stopping rule for Landweber plays the role of the parameter selection rule for Tikhonov. In general, it is rather difficult to construct an appropriate regularization term to obtain the stability for evolutional type inverse problems (see [7]). Therefore, the first advantage of the Landweber iteration suggests to us, in a sense, that we may reconstruct the unknown source on the whole.

Based on the analysis above, we consider another kind of numerical method to recover the unknown coefficient $f(x,t)$ from $g^\delta(x,t)$ directly, i.e., find $\bar{f}(x,t)$ such that

$$J(\bar{f}) = \inf_{f\in\mathcal{B}} J(f),$$

where $\mathcal{B}$ is an appropriate admissible set. Compared with the first numerical method, the second one seems more natural and can be applied for more extensive input data. It seems that this new idea can also be applied to the nonlinear inverse problem which has been discussed in [7], i.e., the determination of the radiative coefficient $p(x,t)$ in the following heat conduction equation:

$$u_t - \Delta u + p(x,t)u = 0, \qquad (x,t) \in Q,$$

from the overspecified data $u(x,t) = g^\delta(x,t)$, $(x,t) \in Q$. We may regard the low-order term pu as an unknown heat source f, i.e., $f = pu$, and recover the source f by the method and then get $p = -f/g^\delta$. Here we shall require $g^\delta(x,t) \neq 0$, $(x,t) \in Q$, while for the case that $g^\delta(x,t)$ may equal zero for some point $(x,t) \in Q$, this method is no longer useful and we shall turn to the TSDS.

This paper is organized as follows. The time semi-discrete scheme is introduced in section 2. In section 3, the integral reconstruction scheme is proposed and the corresponding convergence is proved. Numerical treatments for PDEs (3.13) and (3.14) are given in section 4. Some numerical experiments and results are presented in section 5. Section 6 ends this paper with concluding remarks.

2. Time semi-discrete scheme

Assume that $a(x)$, $b(x)$, $c(x)$ and $\phi(x)$ satisfy

$$a(x), b(x), c(x) \in C^\alpha(0,l), \quad a(x) \geqslant a_0 > 0, \quad c(x) \geqslant 0, \quad \phi(x) \in C^{2,\alpha}(0,l), \tag{2.1}$$

where $\alpha > 0$ and a_0 is a positive constant, and $\phi(x)$ satisfies the homogeneous Dirichlet boundary condition.

The well-known Schauder's theory for parabolic equations (see [10, 16]) guarantees that, for any given coefficient $f(x,t) \in C^{\alpha,\frac{\alpha}{2}}(Q)$, there exists a unique solution $u(x,t) \in C^{2+\alpha,1+\frac{\alpha}{2}}(\bar{Q})$ to equation (1.1).

In this paper, we would not like to discuss the existence and uniqueness of the solution for the inverse problem **P**. Note that $g^\delta(x,t)$, which is obtained from (1.2) or (1.6) by the interpolation and smoothing technique, even if has sufficient smoothness, e.g., $g^\delta \in C^{2,1}(Q)$, is not unique and neither is $f(x,t)$. With regard to the existence, it can easily be seen from (1.4) that for general input data $g^\delta \in L^2(Q)$ the inverse problem has no solution in the sense of classical theory at all. Therefore, it is not so interesting to study these problems. The central issue of this paper is to illustrate the stability of the solution.

To reconstruct the unknown coefficient, we introduce the following time semi-discrete optimal control problem.

Let

$$0 = t_0 < t_1 < t_2 < \cdots < t_N = T$$

be a partition of interval $[0,T]$ with $t_n = n\tau$ and $\tau = \frac{T}{N}$.

Let

$$\mathcal{A} = \{f(x) | |f(x)| \leqslant M, f \in H^1(0,l)\}$$

be the admissible set, where M is a given positive constant.

Beginning with a given function $f_0 \in \mathcal{A}$, we consider the following optimal control problem:

Problem P_n: Assume that $f_0, f_1, \ldots, f_{n-1} \in \mathcal{A}$ are known. Find $f_n \in \mathcal{A}$ such that

$$J_n(f_n) = \min_{f \in \mathcal{A}} J_n(f), \tag{2.2}$$

where

$$J_n(f) = \frac{1}{2}\|u(\cdot, t_n; f) - g^\delta(\cdot, t_n)\|^2_{L^2(0,l)}, \tag{2.3}$$

$u(x, t; f)$ is the solution of (1.1) in $[0, t_n]$ corresponding to the coefficient

$$\tilde{f} = \begin{cases} \frac{t-t_{n-1}}{\tau} f(x) + \frac{t_n - t}{\tau} f_{n-1}(x), & t_{n-1} \leqslant t \leqslant t_n, \\ \frac{t-t_{k-1}}{\tau} f_k(x) + \frac{t_k - t}{\tau} f_{k-1}(x), & t_{k-1} \leqslant t \leqslant t_k, 1 \leqslant k \leqslant n-1. \end{cases} \tag{2.4}$$

With the transformation above, problem **P** is transformed into a sequence of inverse problems $P_n, n = 1, 2, \ldots, N$, which are similar to that in [14], i.e., the unknown coefficient is purely space dependent.

The procedure for the stable reconstruction of the solution u and f can be stated as follows:

Assume that $f_1, f_2, \ldots, f_{n-1}$ have been reconstructed.

Step 1. Choose an initial value of iteration $f = f^0(x)$. For simplicity, we can choose $f^0(x) = 0, x \in (0, l)$.

Step 2. Solve the following initial-boundary value problem:

$$\begin{cases} u_t - a(x)u_{xx} + b(x)u_x + c(x)u = \tilde{f}(x, t), & (x, t) \in (0, l) \times (0, t_n] \\ u(0, t) = u(l, t) = 0, \\ u(x, 0) = \phi(x), \end{cases} \tag{2.5}$$

to obtain the solution $u^0(x, t_n)$, where $\tilde{f}(x, t)$ is as defined in (2.4) with $f = f^0$.

Step 3. Solve the adjoint problem of (2.5)

$$\begin{cases} v_t - (av)_{xx} - (bv)_x + cv = u^0(\cdot, t_n) - g^\delta(\cdot, t_n), & (x, t) \in (0, l) \times (0, t_n] \\ v(0, t) = v(l, t) = 0, \\ v(x, 0) = 0, \end{cases} \tag{2.6}$$

to obtain the solution $v^0(x, t_n)$.

Step 4. Let

$$f^1(x) = f^0(x) - \alpha v^0(x, t_n),$$

where $\alpha > 0$, and let u^1 be the solution of (2.5) with $f = f^1$.

Step 5. Let δ^n be the nth layer noise level, i.e.,

$$\|g^\delta(\cdot, t_n) - g(\cdot, t_n)\|_{L^2(0,l)} \leqslant \delta^n,$$

where $g(x, t)$ is the exact data.

If

$$\|u^1(\cdot, t_n) - g^\delta(\cdot, t_n)\| \leqslant \delta^n,$$

then stop the iteration scheme and take $f = f^1$;

If

$$\|u^1(\cdot, t_n) - g^\delta(\cdot, t_n)\| > \delta^n,$$

then go to *Step 2*. Let $f^1(x)$ be a new initial value of iteration and go on computing by the induction principle.

By continuing the procedure above, we can obtain $f_0, f_1, \ldots, f_N$ and the approximate solution $\tilde{f}(x, t)$.

Remark 2.1. It can easily be seen that the essence of the time semi-discrete scheme is to reconstruct the unknown heat source function layer by layer. The iterative algorithm above is indeed the Landweber–Fridman iteration proposed in [14], where an inverse problem of identifying a stationary source term has been studied carefully. Therefore, the algorithm is stable on every time layer $t = t_n, n = 1, 2, \ldots, N$. In the sense of theoretical analysis, it is natural to consider the smoothness and the stability of the approximate solution $\tilde{f}(x, t)$ in (2.4) as $\tau \to 0$ after $f_1, f_2, \ldots, f_N$ have been reconstructed (see [7]). Such a problem also exists in the numerical computation. Because for a fixed time step size τ, even if

$$\|f_i(x) - \bar{f}_i(x)\|_{L^2[0,l]} \to 0, \qquad i = 1, 2, \ldots, N,$$

where $\bar{f}_i(x)$ is the exact solution on every time layer, one still cannot deduce that

$$\|\tilde{f}(x, t) - \bar{f}(x, t)\|_{L^2(Q)} \to 0.$$

However, if some *a priori* information of the solution is known, e.g., $\bar{f}(x, t) \in C(Q)$, then $\tilde{f}(x, t)$ can be taken as a 'good' approximate solution of $\bar{f}(x, t)$ as τ is relatively small.

3. Integral reconstruction scheme

Undoubtedly, the time semi-discrete scheme mentioned above is an effective method of dealing with the evolutional type inverse problem. However, to reconstruct the unknown source term on every time layer, one has to apply the iteration method to deal with a stationary inverse source problem. For linear inverse problems, we may have some other method. Another kind of natural idea for problem **P** is to reconstruct the unknown heat source on the whole, which is the so-called integral reconstruction scheme (IRS).

We introduce the following optimal control problem:

Problem $\tilde{\mathbf{P}}$: Find $\bar{f}(x, t) \in \mathcal{B}$ such that

$$J(\bar{f}) = \min_{f \in \mathcal{B}} J(f), \tag{3.1}$$

where

$$J(f) = \frac{1}{2} \int_0^T \int_0^l |u(x, t; f) - g^\delta(x, t)|^2 \,\mathrm{d}x \,\mathrm{d}t, \tag{3.2}$$

$$\mathcal{B} = \{f(x, t) | |f| \leqslant M, f \in C^1([0, T], L^2(0, l))\}, \tag{3.3}$$

$u(x, t; f)$ is the solution of (1.1) for a given coefficient $f(x, t) \in \mathcal{B}$.

Let

$$\mathscr{L}u = -a(x)u_{xx} + b(x)u_x + c(x)u$$

and

$$D(\mathscr{L}) = \left\{u | u \in H_0^1(0, l), \mathscr{L}u \in L^2(0, l)\right\}.$$

Theorem 3.1. *(see [11]) Assume that*

$$a(x), b(x) \in C^1[0,l], \qquad c(x) \in C[0,l], \qquad a(x) \geqslant a_0 > 0$$

and

$$\phi(x) \in D(\mathscr{L}), \qquad f(x,t) \in \mathcal{B}.$$

Then the operator $-\mathscr{L}$ *generates a contraction semi-group* $\{S(t)\}_{t\in R^+}$ *in* $L^2(0,l)$ *and there exists a unique solution* $u(x,t) \in C^1([0,T], L^2(0,l)) \bigcap C([0,T], D(\mathscr{L}))$ *to equation (1.1). Furthermore, the solution can be expressed as*

$$u = S(t)\phi + K(t)f, \tag{3.4}$$

where $K(t)$ *is a linear operator defined as*

$$K(t)f = \int_0^t S(t-\tau)f\,\mathrm{d}\tau. \tag{3.5}$$

Using the properties of the contraction semi-group in combination with [17], we have the following lemma:

Lemma 3.2. *Suppose that* $\phi \in L^2(0,l)$ *and* $f \in L^2(Q)$*. Then equation (1.1) has a unique solution* $u \in L^2\big([0,T], H_0^1(0,l)\big) \bigcap C\big([0,T], H_0^1(0,l)\big)$ *in the distributional sense which satisfies*

$$\|u\|_{L^2([0,T],H_0^1(0,l))} \leqslant C(\|f\|_{L^2(Q)} + \|\phi\|_{L^2(0,l)}). \tag{3.6}$$

From theorem 3.1, we can see that finding a solution to the inverse problem is equivalent to solving the following operator equation:

$$K(t)f = g^\delta - S(t)\phi. \tag{3.7}$$

We use the Landweber iteration method (see [9, 15]) to obtain the numerical solution of (3.7). In fact, such a method is the steepest decent algorithm for finding the minimizer of (3.2). Note that (3.7) can be rewritten as

$$f = (I - \alpha K^*K)f + \alpha K^*(g^\delta - S(t)\phi), \tag{3.8}$$

where K^* is the adjoint operator of K, and $\alpha > 0$ is the step size. Then we use the iteration method to solve (3.8), i.e.,

$$\begin{aligned} &f^0 = 0, \\ &f^m = (I - \alpha K^*K)f^{m-1} + \alpha K^*(g^\delta - S(t)\phi), \qquad m = 1, 2, 3, \ldots. \end{aligned} \tag{3.9}$$

From (3.4) and (3.9) we have

$$\begin{aligned} f^m &= f^{m-1} - \alpha K^*(Kf^{m-1} - (g^\delta - S(t)\phi)) \\ &= f^{m-1} - \alpha K^*(u^{m-1} - g^\delta), \end{aligned} \tag{3.10}$$

where u^{m-1} is the solution of (1.1) with $f = f^{m-1}$.

Lemma 3.3. *For any given* $\psi \in C^1([0,T], L^2(0,l))$*, let* $v = K^*\psi$*. Then* v *satisfies the following equation:*

$$\begin{cases} -v_t - (av)_{xx} - (bv)_x + cv = \psi, & (x,t) \in Q \\ v|_{x=0} = v|_{x=l} = 0, \\ v|_{t=T} = 0. \end{cases} \tag{3.11}$$

Proof. By the definition of K, we have

$$K : C^1([0,T], L^2(0,l)) \mapsto C^1([0,T], L^2(0,l)) \bigcap C([0,T], D), \qquad f \mapsto u(f),$$

where $u(f)$ is the solution of the following equation:

$$\begin{cases} u_t - au_{xx} + bu_x + cu = f, & (x,t) \in Q \\ u|_{x=0} = u|_{x=l} = 0, \\ u|_{t=0} = 0. \end{cases} \tag{3.12}$$

Then from (3.11) and (3.12) we have

$$\begin{aligned} \int_0^T \int_0^l (\psi K f - f v)\, \mathrm{d}x\, \mathrm{d}t &= \int_0^T \int_0^l [(-v_t - (av)_{xx} - (bv)_x + cv)u - (u_t - au_{xx} + bu_x + cu)v]\, \mathrm{d}x\, \mathrm{d}t \\ &= -\int_0^1 uv \Big|_{t=0}^{t=T} \mathrm{d}x \\ &= 0, \end{aligned}$$

i.e.,

$$\langle Kf, \psi\rangle = \langle f, v\rangle.$$

By the definition of K^*, we have

$$v = K^*\psi.$$

This completes the proof of lemma 3.3. □

Based on (3.9), (3.10) and lemma 3.3, the procedure for the iteration algorithm can be stated as follows:

Step 1. Choose an initial value of iteration $f = f^0(x,t) \in L^2(Q)$. For simplicity, we can choose $f^0(x,t) = 0$, $(x,t) \in Q$.

Step 2. Solve the following initial-boundary value problem:

$$\begin{cases} u_t - au_{xx} + bu_x + cu = f, & (x,t) \in Q \\ u|_{x=0} = u|_{x=l} = 0, \\ u|_{t=0} = \phi(x), \end{cases} \tag{3.13}$$

to obtain the solution $u^0(x,t)$, where $f = f^0$.

Step 3. Solve the adjoint problem of (3.13)

$$\begin{cases} -v_t - (av)_{xx} - (bv)_x + cv = u^0 - g^\delta, & (x,t) \in Q \\ v|_{x=0} = v|_{x=l} = 0, \\ v|_{t=T} = 0, \end{cases} \tag{3.14}$$

to obtain the solution $v^0(x,t)$.

Step 4. Let

$$f^1 = f^0 - \alpha v^0,$$

where $\alpha > 0$, and let u^1 be the solution of (3.13) with $f = f^1$.

Step 5. Let δ be the noise level, i.e.,

$$\|g^\delta - g\|_{L^2(Q)} \leqslant \delta,$$

where $g(x,t)$ is the exact input data.

If

$$\|u^1 - g^\delta\| \leqslant \delta,$$

then stop the iteration scheme and take $f = f^1$;

Otherwise go to *Step 2*. Let $f^1(x,t)$ be the new initial value of iteration and go on computing by the induction principle.

If the input data g are a 'real' temperature distribution, i.e., g is the solution of equation (1.1) with a given heat source $f \in L^2(Q)$, then we have the following convergence results:

Theorem 3.4. *Let $\phi \in L^2(0,l)$ and let $g \in L^2\big([0,T], H_0^1(0,l)\big) \bigcap C\big([0,T], H_0^1(0,l)\big)$ be the input data. Assume that α satisfies $0 < \alpha < 1/\|K\|^2$. Let u^k be the kth approximation with g^δ replaced by g in the iterative procedure above. Then we have*

$$\lim_{k\to\infty} \|u^k - g\|_{L^2([0,T],H_0^1(0,l))} = 0 \tag{3.15}$$

for any initial guess $f^0 \in L^2(Q)$.

Proof. From theorem 3.1, lemma 3.3 and the iterative procedure given above, we have

$$\begin{aligned} f^{k+1} &= f^k - \alpha v^k \\ &= f^k - \alpha K^*(u^k - g) \\ &= f^k - \alpha K^*(Kf^k - (g - S\phi)). \end{aligned}$$

From the assumption $0 < \alpha < 1/\|K\|^2$ and the standard theory for the Landweber iteration (see [9]), we have that the sequence f^k converges to f in $L^2(Q)$. Then from lemma 3.2 one can easily obtain that u^k converges to g in $L^2\big([0,T], H_0^1(0,l)\big)$.

This completes the proof of theorem 3.4. □

4. Numerical schemes for PDEs (3.13) and (3.14)

In this paper, we use the finite difference method to solve the PDEs (3.13) and (3.14). Since the explicit difference scheme is conditionally stable, the implicit scheme which is absolutely stable is employed to obtain the numerical solution.

The numerical scheme for (3.13) is standard and is thus omitted. For the backward parabolic equation (3.14), we make the following change in the variable:

$$\tau = T - t.$$

Then (3.14) is transformed into the following forward parabolic equation:

$$\begin{cases} v_\tau - (av)_{xx} - (bv)_x + cv = u^0(\cdot, T-\tau) - g^\delta(\cdot, T-\tau), & (x,t) \in Q, \\ v|_{x=0} = v|_{x=l} = 0, \\ v|_{\tau=0} = 0. \end{cases} \tag{4.1}$$

So the numerical method for (3.13) can also be applied to (4.1). It should be pointed out that one must be very careful about the change of variable in (4.1), namely that after the numerical solution of (4.1) has been obtained, the variable τ should be changed back into the variable t, which is different from the numerical procedure of (2.6). Equation (3.14) can also be solved without any change of variable.

For the sake of simplicity, we assume that

$$a = 1, \qquad b = c = 0,$$

in (3.14). Assume that the domain $Q = [0, l] \times [0, T]$ is divided into a $J \times N$ mesh with the spatial step size $h = \frac{l}{J}$ in the x-direction and the time step size $k = \frac{T}{N}$.

Grid points (x_j, t_n) are defined by

$$x_j = jh, \qquad j = 0, 1, 2, \ldots, J,$$

$$t_n = nk, \qquad n = 0, 1, 2, \ldots, N,$$

in which J and N are two integers. The notation v_j^n is used for the finite difference approximation of $v(jh, nk)$.

Using the final condition

$$v(x, T) = 0, \qquad 0 \leqslant x \leqslant l,$$

equation (3.14) is solved approximately, commencing with final values

$$v_j^N = 0, \qquad j = 0, 1, 2, \ldots, J, \tag{4.2}$$

and boundary values

$$v_0^n = 0, \qquad n = 0, 1, 2, \ldots, N-1, \tag{4.3}$$

$$v_J^n = 0, \qquad n = 0, 1, 2, \ldots, N-1. \tag{4.4}$$

The implicit difference scheme leads to the following difference equation for (3.14):

$$-\frac{v_j^{n+1} - v_j^n}{k} - \frac{v_{j+1}^n - 2v_j^n + v_{j-1}^n}{h^2} = F_j^n, \tag{4.5}$$

for $1 \leqslant j \leqslant J-1$, and $0 \leqslant n \leqslant N-1$, where

$$F_j^n = (u^0 - g^\delta)_j^n.$$

This method is stable in maximum-norm without any restriction on k and h, and the truncation error is $O(k + h^2)$.

Remark 4.1. To improve the precision, one may use the so-called Crank–Nicolson scheme to replace the old one. In such a case, the difference equation can be written as

$$-\frac{v_j^{n+1} - v_j^n}{k} - \frac{\lambda}{2}\left[\left(v_{j+1}^{n+1} - 2v_j^{n+1} + v_{j-1}^{n+1}\right) + \left(v_{j+1}^n - 2v_j^n + v_{j-1}^n\right)\right] = 0,$$

whose truncation error is $O(k^2 + h^2)$. However, we find that it is not necessary to do so because the implicit scheme (4.5) is enough to meet the needs, as can be seen from the following numerical experiments.

5. Numerical experiments and results

We have performed three numerical experiments to test the stability of our algorithm. In all experiments, some basic parameters are

$$l = T = 1, \qquad a(x) \equiv 1, \qquad b(x) = c(x) \equiv 0, \qquad k = h = 0.01.$$

We use the symbol σ to denote the stopping parameter in the iteration procedure, i.e.,

$$\sigma = \|u(x, t; f) - g^\delta(x, t)\|_{L^2(Q)},$$

and the symbols $\mathcal{E}$ and $\mathcal{E}_1$ to denote the absolute and relative L^2-norm error between the exact solution $f(x, t)$ to be identified and the numerically reconstructed solution $\tilde{f}(x, t)$, i.e.,

$$\mathcal{E} = \|\tilde{f}(x, t) - f(x, t)\|_{L^2(Q)},$$

$$\mathcal{E}_1 = \|\tilde{f}(x, t) - f(x, t)\|_{L^2(Q)} / \|f(x, t)\|_{L^2(Q)}.$$

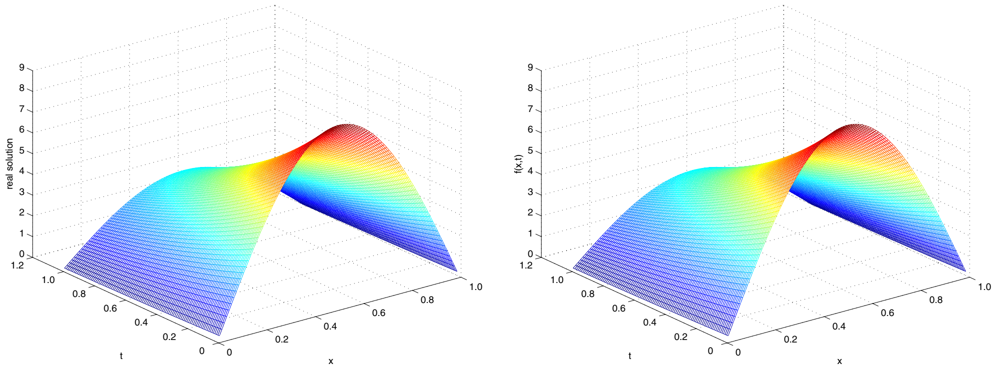

Figure 1. Reconstruction of the continuous heat source by the TSDS.

Example 1. In the first numerical experiment, we take

$$\phi(x) = \sin(\pi x), \qquad x \in [0, 1],$$
$$f(x,t) = (\pi^2 - 1)\exp(-t)\sin(\pi x), \qquad (x,t) \in [0,1] \times [0,1].$$

In this case, the direct problem (1.1) has the following analytical solution:

$$u(x,t) = \exp(-t)\sin(\pi x), \qquad (x,t) \in [0,1] \times [0,1],$$

and thus the extra observation is given by

$$g(x,t) = u(x,t), \qquad (x,t) \in [0,1] \times [0,1].$$

For the first kind of numerical method (TSDS), we know that the initial value $f(x, 0)$ cannot be recovered. So we take the initial value as the exact solution simply for convenience. The exact solution and the reconstructed one are shown in figure 1, where the iteration parameter is taken as $\alpha = 100$. The initial guess is taken to be zero on every layer. Clearly, this initial guess is not good at all, but the TSDS converges very stably and fast (the result shown is obtained from the 100th iteration and the CPU time is 8.984 s) and the reconstruction solution seems to be very satisfactory (the maximal error $\|\tilde{f} - f\|_{C((0,l)\times(0,T))}$ is less than 8×10^{-3}).

For the second kind of numerical method (IRS), we know that the terminal value $f(x, T)$ cannot be recovered, which can easily be seen from steps 3 and 4 of the algorithm. During the iteration procedure, it always remains at zero. The reconstruction results are shown in figure 2, where the iteration parameter is also taken as $\alpha = 100$. We can see from this figure that the unknown source function $f(x,t)$ can be recovered very well (the maximal error $\|\tilde{f} - f\|_{C((0,l)\times(0,T))}$ is less than 6×10^{-3}). Moreover, the IRS converges faster than the TSDS (the CPU time is 3.532 s).

If we apply the noisy data generated in the form

$$g^\delta(x,t) = g(x,t)[1 + \delta \times \text{random(x)}]$$

with $\delta = 5\%$, the reconstruction results obtained by the IRS are also satisfactory ($\mathcal{E} = 0.2673, \mathcal{E}_1 = 0.0648$); see figure 3. We can see from this figure that after 150 iterations (denoted by k) the stopping parameter σ is 0.0050 which is much less than δ. The numerical results obtained by the TSDS are similar to those of the IRS and thus are omitted. *(To save space, we only present the numerical results obtained by the IRS in the following test examples.)*

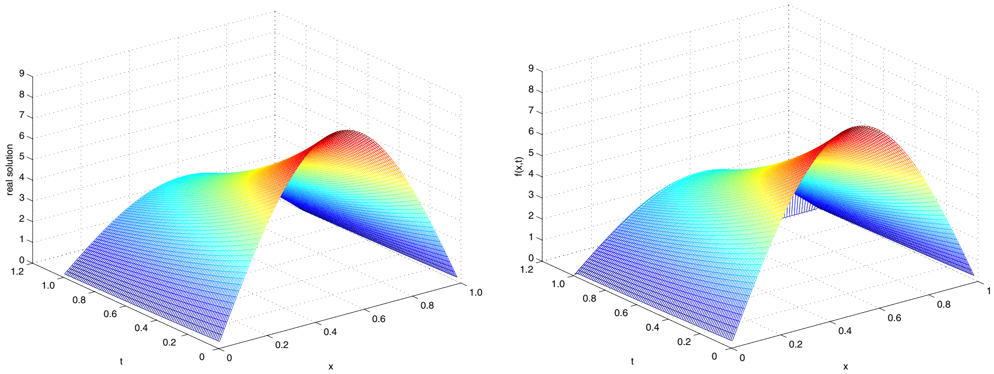

Figure 2. Reconstruction of the continuous heat source by the IRS.

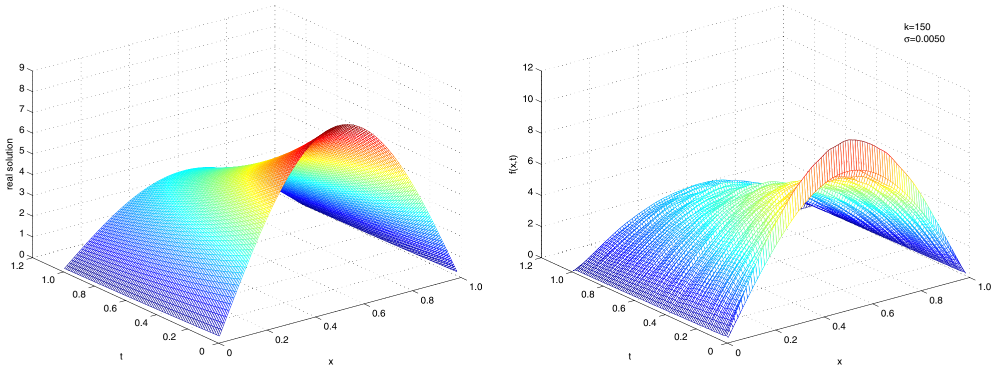

Figure 3. Reconstruction of the continuous heat source with noisy data.

Remark 5.1. For the TSDS, the initial guess of every layer is taken as $f_n = 0, n = 1, 2, \ldots, N$. But it is not essential to do so. When f_n is to be computed, one may take f_{n-1} as a starting value for the iterative procedure. We find that as $\alpha = 100$, to obtain similar accuracy ($\|\tilde{f} - f\|_{C((0,l)\times(0,T))} = 0.0068$), the required CPU time is 5.7030 s which is still longer than that of the IRS. However, when the value of α becomes larger, the CPU time becomes less. As $\alpha = 10\,000$, the CPU time needed is 0.0460 s. This may be the most suitable value of α. When $0 < \alpha \leqslant 10\,000$, the bigger the α is the more quickly the algorithm converges. The maximum of α can be taken to be 23 960, which is far bigger than that of the IRS (see remark 5.2).

Example 2. In the second numerical experiment, we take

$$f(x,t) = \begin{cases} 2(\pi^2 - 1)x\exp(-t), & 0 \leqslant x \leqslant 0.5, \\ 2(\pi^2 - 1)(1 - x)\exp(-t), & 0.5 \leqslant x \leqslant 1, \end{cases} \tag{5.1}$$

and $\phi(x)$ is the same as that of the first experiment. It can easily be seen that $f(x,t)$ is a continuous rather than differentiable function. Being different from example 1, the direct problem (1.1) has no analytical solution. So the observation data $g(x,t)$ are given with the numerical form, i.e.,

$$g(x,t) = u(x,t; f),$$

where $u(x,t; f)$ is the numerical solution of (1.1) with the input source (5.1).

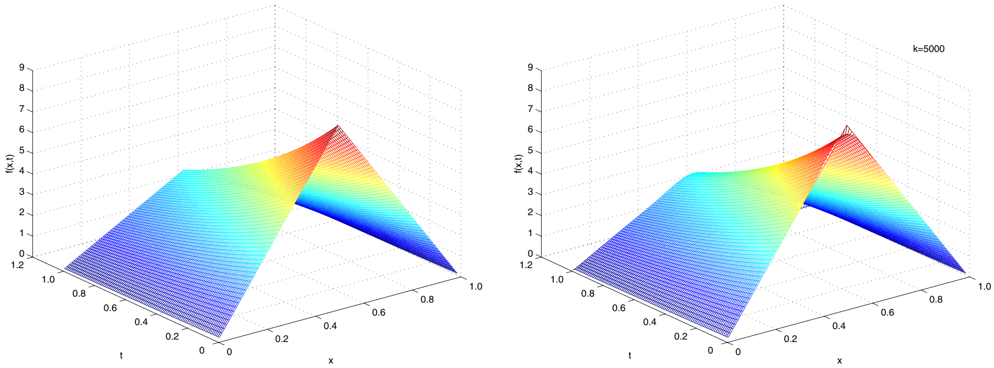

Figure 4. Reconstruction of the heat source $f(x,t) \in C^0 \backslash C^1(Q)$ by the IRS.

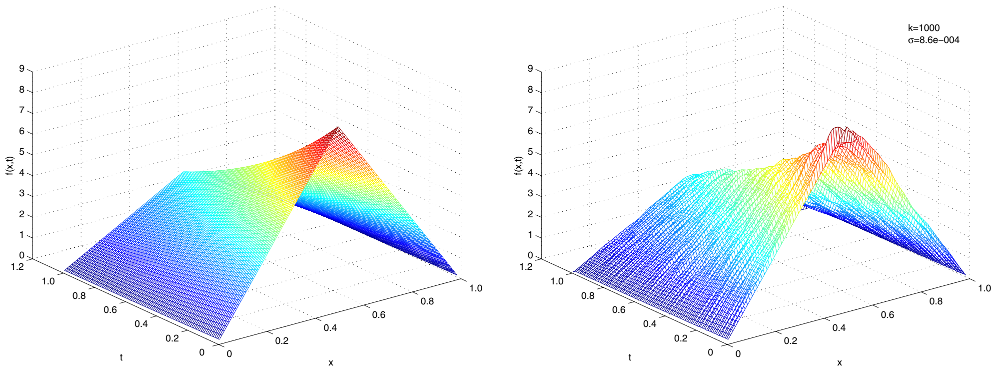

Figure 5. Reconstruction of the heat source $f(x,t) \in C^0 \backslash C^1(Q)$ with the noisy data.

Table 1. Numerical comparisons at some points (x_j, t_n) from exact data for example 2.

(j, n)	$f(25, n)$	$\tilde{f}(25, n)$	$f(50, n)$	$\tilde{f}(50, n)$	$f(75, n)$	$\tilde{f}(75, n)$
$n = 20$	3.6309	3.6372	7.2618	7.0047	3.6309	3.6372
$n = 50$	2.6898	2.6945	5.3797	5.1892	2.6898	2.6945
$n = 80$	1.9927	1.9962	3.9854	3.8443	1.9927	1.9962

The exact solution together with the recovery one is shown in figure 4, and the numerical results at some points are given in table 1, where the nodal number (j, n) corresponds to the point

$$(x_j, t_n) = (jh, nk) \in Q,$$

while $f(j, n) := f(x_j, t_n)$ and $\tilde{f}(j, n) := \tilde{f}(x_j, t_n)$ represent the exact solution and the inversion one, respectively. We can see that the cuspidal line of $f(x, t)$ is recovered very well after 5000 iterations ($\mathcal{E} = 0.1369$, $\mathcal{E}_1 = 0.0407$). Here we take the initial value $f(x, 0)$ as the exact solution simply for the convenience of the reader's comparison.

Then, we also consider the noisy data by taking $\delta = 1\%$, while the other parameters are kept unchanged. The inversion performance is given in figure 5 and table 2. It

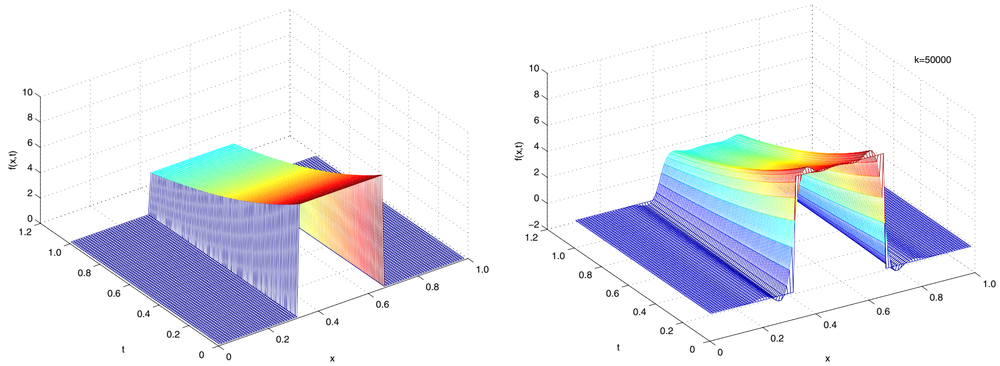

Figure 6. Reconstruction of the discontinuous heat source by the IRS.

Table 2. Numerical comparisons at some points (x_j, t_n) from noisy data for example 2.

(j, n)	$f(25, n)$	$\tilde{f}(25, n)$	$f(50, n)$	$\tilde{f}(50, n)$	$f(75, n)$	$\tilde{f}(75, n)$
$n = 20$	3.6309	3.5620	7.2618	6.8040	3.6309	3.6803
$n = 50$	2.6898	2.6678	5.3797	5.0779	2.6898	2.4812
$n = 80$	1.9927	2.0231	3.9854	3.8454	1.9927	1.9553

Table 3. Numerical comparisons at some points (x_j, t_n) from exact data for example 3.

(j, n)	$f(25, n)$	$\tilde{f}(25, n)$	$f(50, n)$	$\tilde{f}(50, n)$	$f(75, n)$	$\tilde{f}(75, n)$
$n = 20$	0	−0.1494	7.2618	7.3160	0	−0.0250
$n = 50$	0	−0.1107	5.3797	5.4199	0	−0.0185
$n = 80$	0	−0.0820	3.9854	4.0150	0	−0.0137

can be seen that the reconstructed solution matches the exact one very satisfactorily ($\mathcal{E} = 0.1744, \mathcal{E}_1 = 0.0518$).

Example 3. In the third numerical experiment, we take

$$f(x, t) = \begin{cases} (\pi^2 - 1)\exp(-t), & (x, t) \in \left[\frac{1}{3}, \frac{2}{3}\right] \times [0, 1], \\ 0, & (x, t) \in Q \Big\backslash \left[\frac{1}{3}, \frac{2}{3}\right] \times [0, 1], \end{cases} \tag{5.2}$$

and $\phi(x)$ is the same as that of the first experiment. The heat source $f(x, t)$ is a discontinuous function and thus the direct problem (1.1) has no analytical solution.

The exact solution and the recovery one are shown in figure 6, and the numerical results at some points are given in table 3. We can see that the discontinuous property of $f(x, t)$ is recovered very well after 50 000 iterations ($\mathcal{E} = 0.5330, \mathcal{E}_1 = 0.1567$). Noticing the poor smoothness of $f(x, t)$, we need many more iterations than in experiments 1 and 2. Similarly, the reconstruction of $f(x, t)$ from the noisy data $g^\delta(x, t)$ is also performed, where the noise level δ is also taken as 1%. The results are shown in figure 7 and table 4 ($\mathcal{E} = 0.7397, \mathcal{E}_1 = 0.2175$).

Remark 5.2. From theorems 3.1 and 3.4 we know that the iterative procedure is convergent for $0 < \alpha \leqslant 1/\|K\|^2$, where $\|K\| \leqslant 1$. It should be mentioned that the regularization

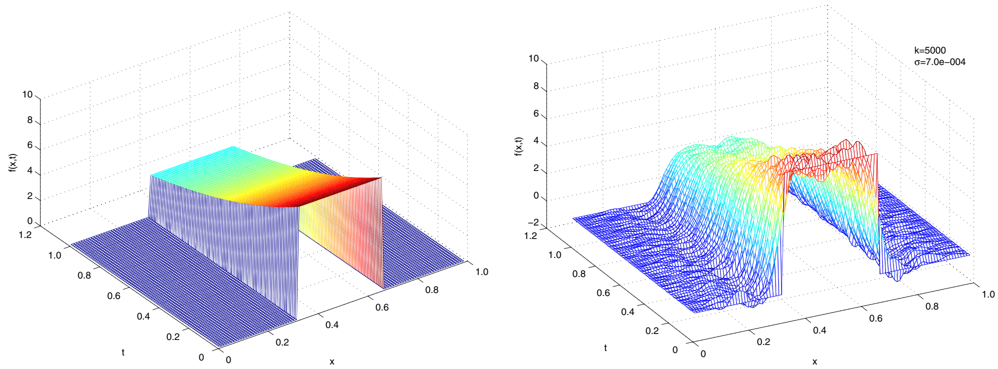

Figure 7. Reconstruction of the discontinuous heat source with the noisy data.

Table 4. Numerical comparisons at some points (x_j, t_n) from noisy data for example 3.

(j, n)	$f(25, n)$	$\tilde{f}(25, n)$	$f(50, n)$	$\tilde{f}(50, n)$	$f(75, n)$	$\tilde{f}(75, n)$
$n = 20$	0	−0.4753	7.2618	8.1287	0	−0.6020
$n = 50$	0	−0.1931	5.3797	5.4015	0	−0.5035
$n = 80$	0	−0.0940	3.9854	3.9400	0	−0.0479

parameter α plays a major role in the numerical simulation of the inverse problem. During the course of numerical computation, we find that the maximum of α can be taken as 210 and as $0 < \alpha \leqslant 210$, the bigger the α is, the more quickly the algorithm converges. If the parameter exceeds this range, the iterative procedure will diverge.

Moreover, the initial guess $f^0(x, t)$ is taken as zero in the numerical computation. If we take f^0 as some other value, e.g., $f^0 \equiv 1, (x, t) \in Q$, then the unknown heat source can also be recovered well in the internal part of the domain except for the two boundaries $x = 0$ and $x = 1$.

6. Concluding remarks

In this paper, we solve the inverse problem **P** of recovering the heat source coefficient $f(x, t)$ in the following heat conduction equation:

$$u_t - a(x)u_{xx} + b(x)u_x + c(x)u = f(x, t)$$

in an optimal control framework. Such a problem is a natural extension of that in [14]. Being different from [14], the problem discussed in this paper contains two independent variables x and t, which is often known as the evolutional inverse problem in mathematics. Motivated by the idea in [7, 14], the time semi-discrete scheme (TSDS) is applied to recover the unknown source layer by layer. For ordinary input data $g^\delta \in C([0, T], L^2(0, l))$, this method is an efficient tool to deal with the inverse problem. The key ideology of such a method is to transform the problem **P** into a sequence of stationary inverse source problems. When the parameter α is taken as 10 000, the iterative algorithm of the TSDS converges very quickly. Another kind of numerical method which is the so-called integral reconstruction scheme (IRS) is also proposed in this paper. Compared with the TSDS, the IRS can be applied for more extensive input data, but may need a few more computations. Both theoretical and numerical studies have been provided. Numerical experiments show that the two numerical algorithms designed in this paper are stable, and the heat source $f(x,t)$ is recovered very well.

Moreover, as mentioned in section 1, the inverse problem **P** is indeed a numerical differential problem with noisy input data (see, e.g., [18, 12]). So this paper also provides a new idea, in a sense, to deal with numerical differential problems with regard to the second-order parabolic differential operator.

Acknowledgments

We would like to thank the anonymous referees for their valuable comments and helpful suggestions to improve the earlier version of this paper. We also wish to acknowledge conversations with Dr Nan Gu of the Department of Applied Mathematics at the University of Purdue (USA) regarding this problem. This work is supported by NNSF of China (No 10572051) and NSF of Gansu province of China (No 3ZS051-A25-030) and Optional Subject Funds by Lanzhou Jiaotong University (No 409043).

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